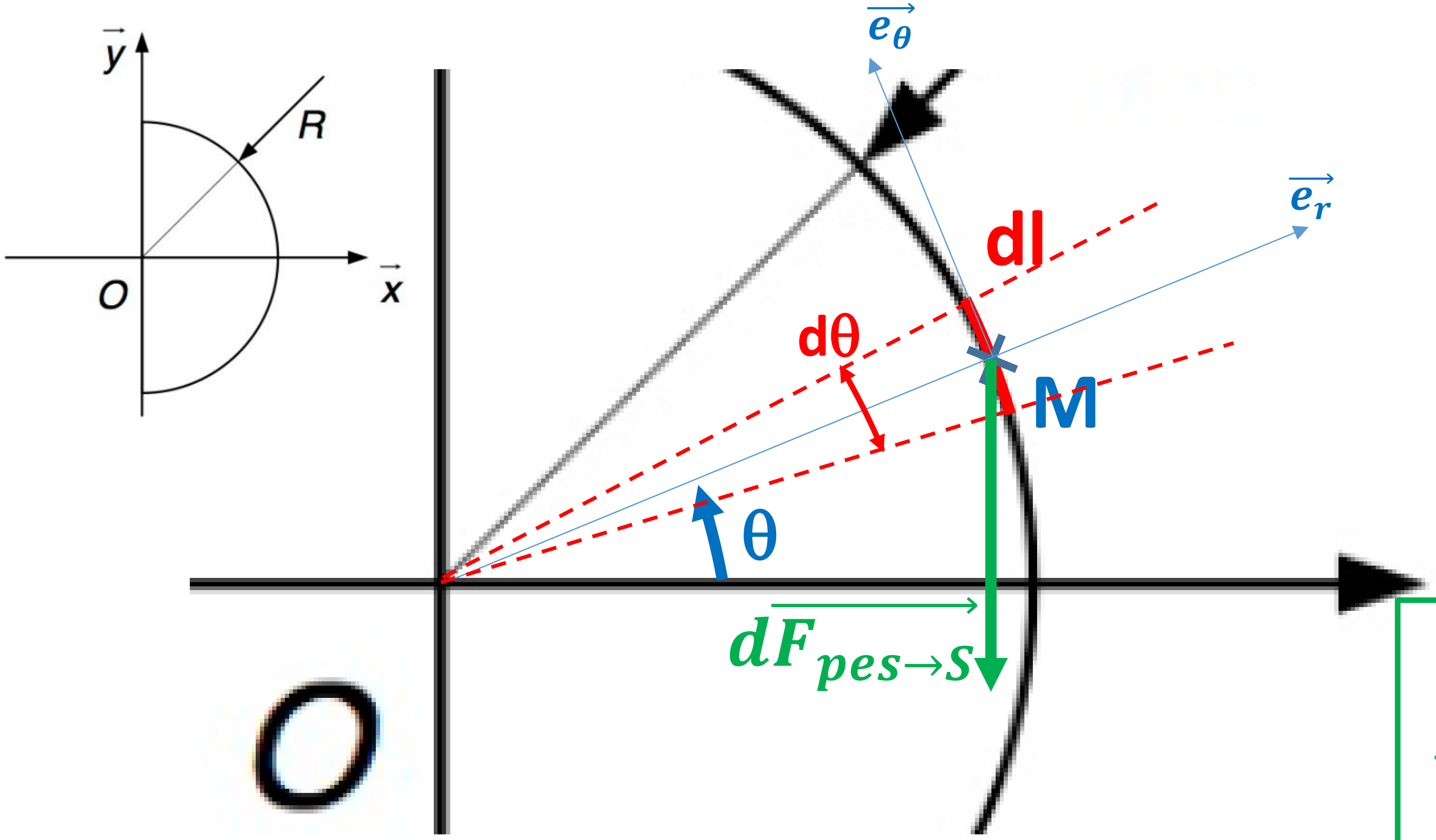
Lycée Fermat Toulouse

Déterminer les lois de commande en effort d'un mécanisme

MODÉLISATION DES ACTIONS MÉCANIQUES
PRINCIPE FONDAMENTAL DE LA STATIQUE

MPSI/PCSI

EXERCICE 1 : Modélisation de l'action de pesanteur



$$\overrightarrow{OM} = R.\overrightarrow{e_r}$$

$$dl = R.d\theta$$

$$\overrightarrow{dF_{pes\to S}} = \lambda\vec{g}dl$$

λ : masse linéique [kg.m^{-1}]

$$\{T_{pes\to S}\} = \begin{Bmatrix} \overrightarrow{R_{pes\to S}} \\ \overrightarrow{M_{O,pes\to S}} \end{Bmatrix}$$

EXERCICE 1 : Modélisation de l'action de pesanteur

$$\overrightarrow{R_{pes\to S}} = \int_S \overrightarrow{dF_{pes\to S}} = \int_S \lambda\vec{g}\,dl$$

$$\overrightarrow{R_{pes\to S}} = -\lambda g R.\vec{y}\int_S d\theta$$

$$\overrightarrow{R_{pes\to S}} = -\lambda g R.\vec{y}\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} d\theta$$

$$\boxed{\overrightarrow{R_{pes\to S}} = -\lambda g R\pi.\vec{y}}$$

$$\overrightarrow{OM} = R.\overrightarrow{e_r}$$

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EXERCICE 1 : Modélisation de l'action de pesanteur

$$\overrightarrow{M_{O,pes\to S}} = \int_S \overrightarrow{OM} \wedge \overrightarrow{dF_{pes\to S}} = \int_S R.\overrightarrow{e_r} \wedge \lambda\vec{g}dl$$

$$\overrightarrow{OM} = R.\overrightarrow{e_r}$$

dl = R.dθ

$$\overrightarrow{dF_{pes\to S}} = \lambda\vec{g}dl$$

λ : masse linéique [kg.m^{-1}]

$$\overrightarrow{M_{O,pes\to S}} = -\lambda g R^2 \int_S \overrightarrow{e_r} \wedge \vec{y}.d\theta$$

$$\overrightarrow{M_{O,pes\to S}} = -\lambda g R^2.\vec{z}\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} cos\theta d\theta$$

$$\boxed{\overrightarrow{M_{O,pes\to S}} = -2\lambda g R^2.\vec{z}}$$

$$\boxed{\overrightarrow{R_{pes\to S}} = -\lambda g R\pi.\vec{y}}$$

$$\boxed{\{T_{pes\to S}\} = \begin{Bmatrix} \overrightarrow{R_{pes\to S}} \\ \overrightarrow{M_{O,pes\to S}} \end{Bmatrix}}$$

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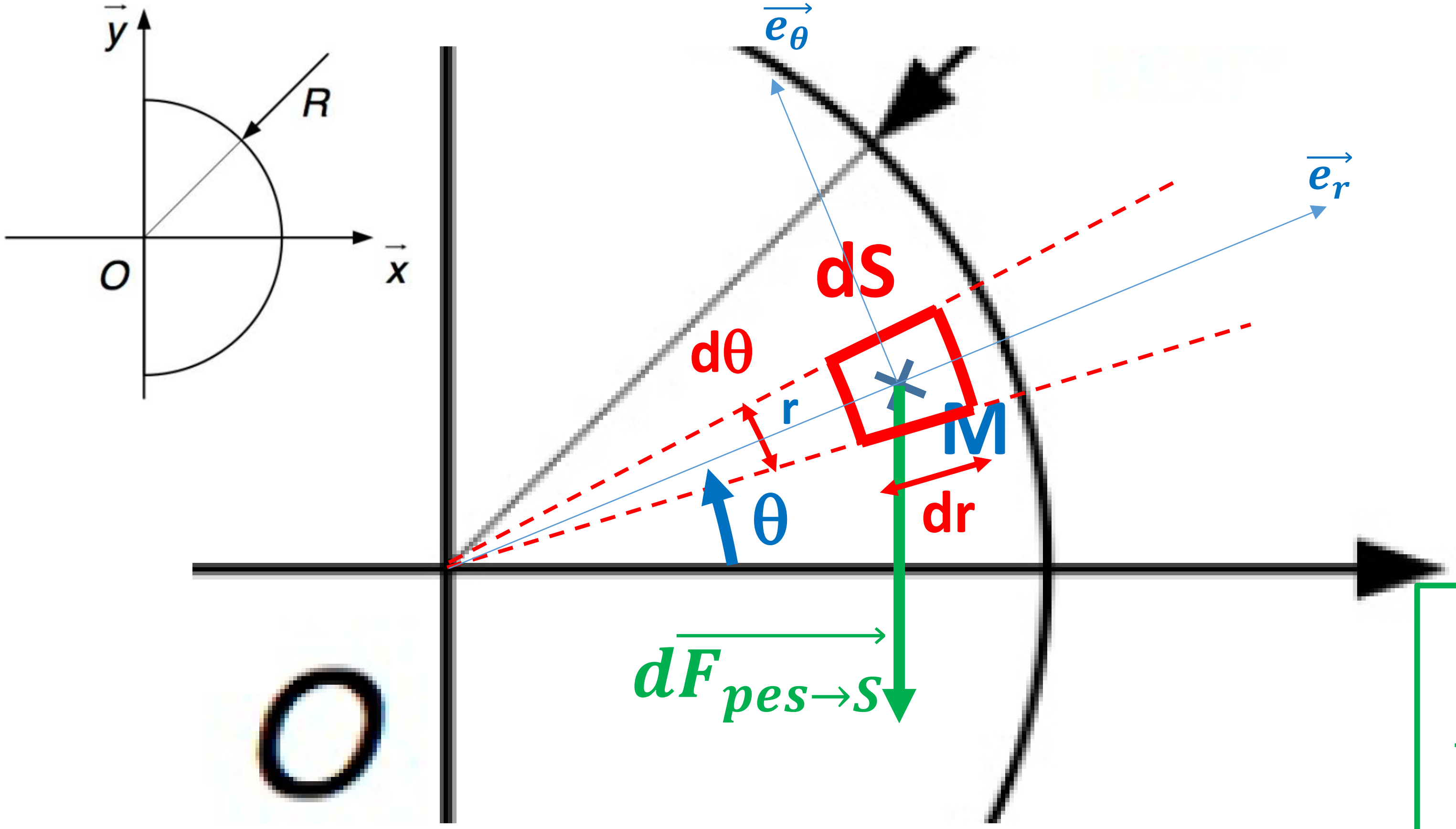

$$\overrightarrow{OM} = r.\vec{e_r}$$

$$dS = r.dr.d\theta$$

$$\overrightarrow{dF_{pes\to S}} = \sigma\vec{g}dS$$

σ : masse surfacique [kg.m^{-2}]

$$\{T_{pes\to S}\} = \begin{Bmatrix} \overrightarrow{R_{pes\to S}} \\ \overrightarrow{M_{O,pes\to S}} \end{Bmatrix}$$

EXERCICE 1 : Modélisation de l'action de pesanteur

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$$dS = r.dr.d\theta$$

$$\overrightarrow{dF_{pes\to S}} = \sigma\vec{g}dS$$

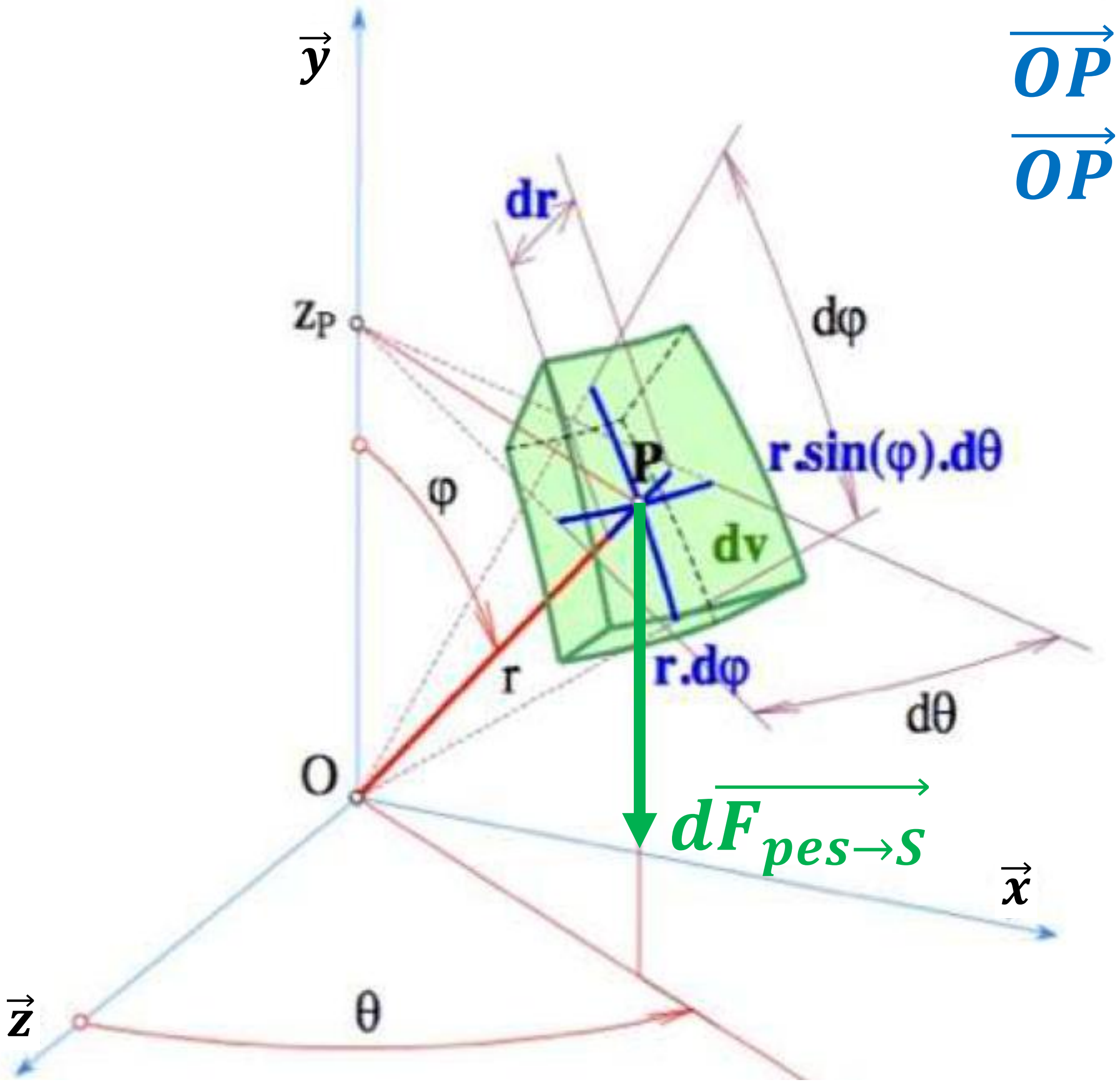
σ : masse surfacique [kg.m^{-2}]

$$\overrightarrow{R_{pes\to S}} = -\sigma g\frac{R^2}{2}\pi.\vec{y}$$

$$\overrightarrow{M_{O,pes\to S}} = -2\sigma g\frac{R^3}{3}.\vec{z}$$

$$\{T_{pes\to S}\} = \begin{Bmatrix} \overrightarrow{R_{pes\to S}} \\ \overrightarrow{M_{O,pes\to S}} \end{Bmatrix}$$

EXERCICE 1 : Modélisation de l'action de pesanteur



$$\overrightarrow{OP} = r.\overrightarrow{e_r}$$

$$\overrightarrow{OP} = r\sin\varphi\sin\theta.\vec{x} + r\cos\varphi.\vec{y} + r\sin\varphi\cos\theta.\vec{z}$$

$$\overrightarrow{dF_{pes\to S}} = \rho\vec{g}dV = -\rho g\vec{y}dV$$

ρ : masse volumique [kg.m^{-3}]

$$dV = r^2\sin\varphi.dr.d\theta.d\varphi$$

$$\{T_{pes\to S}\} = \begin{Bmatrix} \overrightarrow{R_{pes\to S}} \\ \overrightarrow{M_{O,pes\to S}} \end{Bmatrix}$$

EXERCICE 1 : Modélisation de l'action de pesanteur

$$\overrightarrow{R}_{pes\rightarrow S} = \int_S \overrightarrow{dF_{pes\rightarrow S}}$$

$$\overrightarrow{R}_{pes\rightarrow S} = -\rho g.\overrightarrow{y}\int_S dV$$

$$\boxed{\overrightarrow{R}_{pes\rightarrow S} = -\rho g\frac{2}{3}\pi R^3.\overrightarrow{y}}$$

$$\overrightarrow{OP} = r\sin\varphi\sin\theta.\overrightarrow{x} + r\cos\varphi.\overrightarrow{y} + r\sin\varphi\cos\theta.\overrightarrow{z}$$

$$dV = r^2\sin\varphi.dr.d\theta.d\varphi$$

$$\overrightarrow{dF_{pes\rightarrow S}} = -\rho g\overrightarrow{y}dV$$

$$\boxed{\{T_{pes\rightarrow S}\} = \begin{Bmatrix} \overrightarrow{R}_{pes\rightarrow S} \\ \overrightarrow{M}_{O,pes\rightarrow S} \end{Bmatrix}}$$

$\overrightarrow{OP} = r\sin\varphi\sin\theta.\vec{x} + r\cos\varphi.\vec{y} + r\sin\varphi\cos\theta.\vec{z}$

EXERCICE 1 : Modélisation de l'action de pesanteur

$dV = r^2\sin\varphi.dr.d\theta.d\varphi$

$\overrightarrow{dF_{pes\to S}} = -\rho g\vec{y}dV$

$$\overrightarrow{M_{O,pes\to S}} = \int_S \overrightarrow{OP} \wedge \overrightarrow{dF_{pes\to S}} = \int_S \overrightarrow{OP} \wedge -\rho g\vec{y}.dV$$

$$\overrightarrow{M_{O,pes\to S}} = -\rho g \int_S (r\sin\varphi\sin\theta.\vec{x} + r\sin\varphi\cos\theta.\vec{z}) \wedge \vec{y}.dV$$

$$\overrightarrow{M_{O,pes\to S}} = -\rho g \int_S (r\sin\varphi\sin\theta.\vec{z} - r\sin\varphi\cos\theta.\vec{x}).dV$$

$$\overrightarrow{M_{O,pes\to S}} = -\rho g \int_{r=0}^{R}\int_{\varphi=0}^{\pi}\int_{\theta=0}^{\pi} r^2\sin\varphi(r\sin\varphi\sin\theta.\vec{z} - r\sin\varphi\cos\theta.\vec{x}).drd\theta d\varphi$$

$$\overrightarrow{M_{O,pes\to S}} = -\rho g \int_{r=0}^{R}\int_{\varphi=0}^{\pi}\int_{\theta=0}^{\pi} r^2\sin\varphi(r\sin\varphi\sin\theta.\vec{z}).drd\theta d\varphi$$

$$\overrightarrow{OP} = r\sin\varphi\sin\theta.\vec{x} + r\cos\varphi.\vec{y} + r\sin\varphi\cos\theta.\vec{z}$$

EXERCICE 1 : Modélisation de l'action de pesanteur

$$dV = r^2\sin\varphi.dr.d\theta.d\varphi$$

$$\overrightarrow{dF_{pes\rightarrow S}} = -\rho g\vec{y}dV$$

$$\overrightarrow{M_{O,pes\rightarrow S}} = -\rho g \int_{r=0}^{R}\int_{\varphi=0}^{\pi}\int_{\theta=0}^{\pi} r^3\sin^2\varphi\sin\theta.\vec{z}.drd\theta d\varphi$$

$$\overrightarrow{M_{O,pes\rightarrow S}} = -\rho g \int_{r=0}^{R} r^3 dr \int_{\varphi=0}^{\pi}\sin^2\varphi d\varphi \int_{\theta=0}^{\pi}\sin\theta d\theta.\vec{z}$$

$$\boxed{\overrightarrow{M_{O,pes\rightarrow S}} = -\pi\rho g\frac{R^4}{4}.\vec{z}}$$

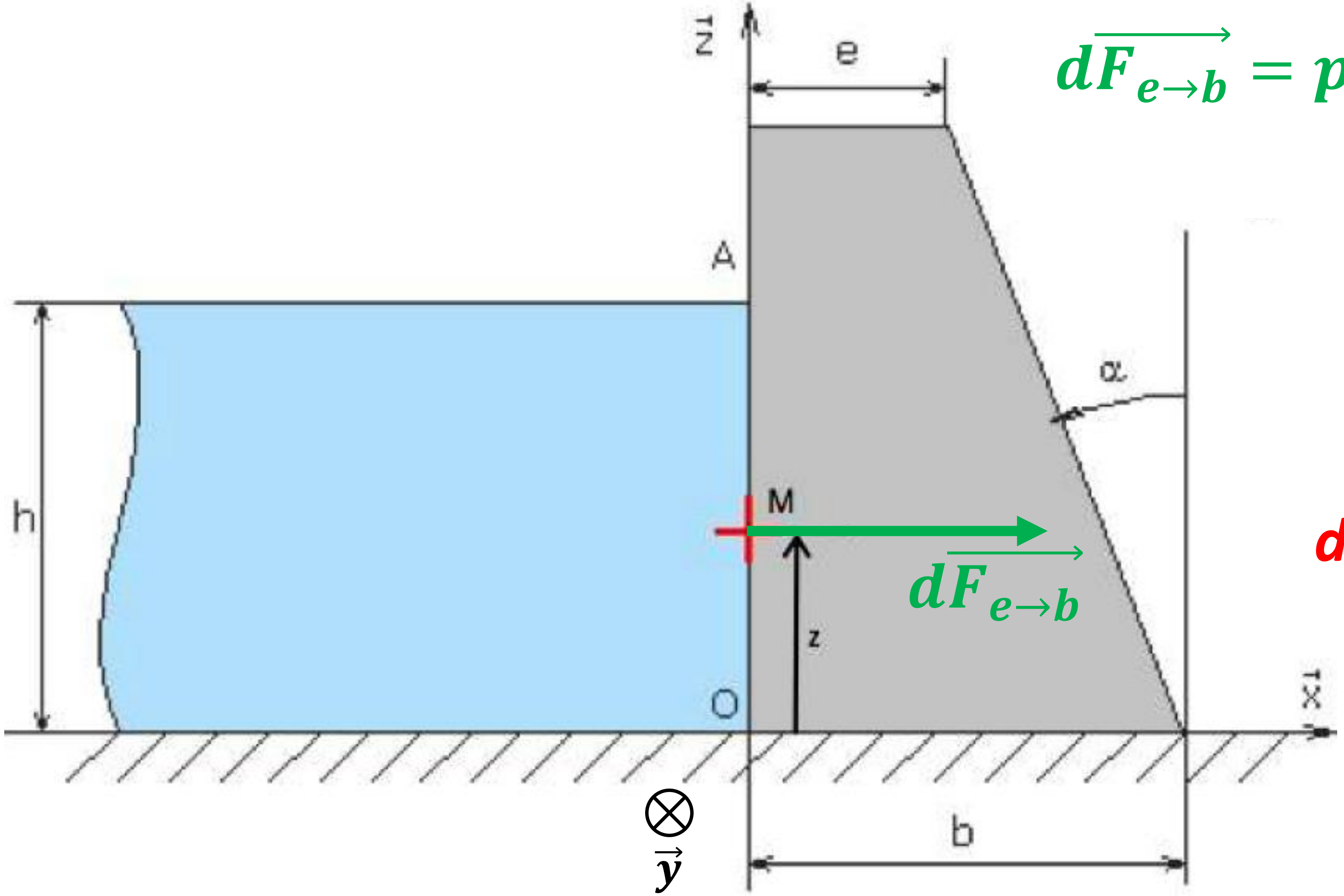
NB :

$$\boxed{\{T_{pes\rightarrow S}\} = \begin{Bmatrix} -\rho g\frac{2}{3}\pi R^3.\vec{y} \\ -\pi\rho g\frac{R^4}{4}.\vec{z} \end{Bmatrix}_O}$$

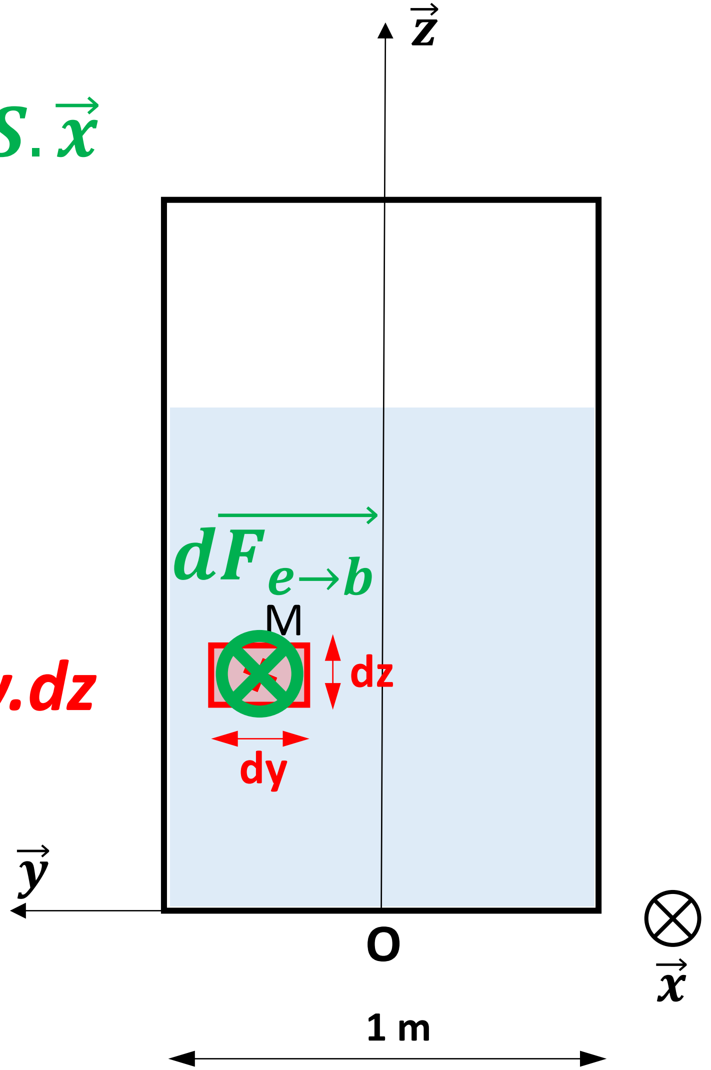
$$\boxed{\{T_{pes\rightarrow S}\} = \begin{Bmatrix} -\rho g\frac{2}{3}\pi R^3.\vec{y} \\ \vec{0} \end{Bmatrix}_G}$$

EXERCICE 2 : Barrage pesant

$$\overrightarrow{dF_{e\rightarrow b}} = p(z)dS.\vec{x}$$



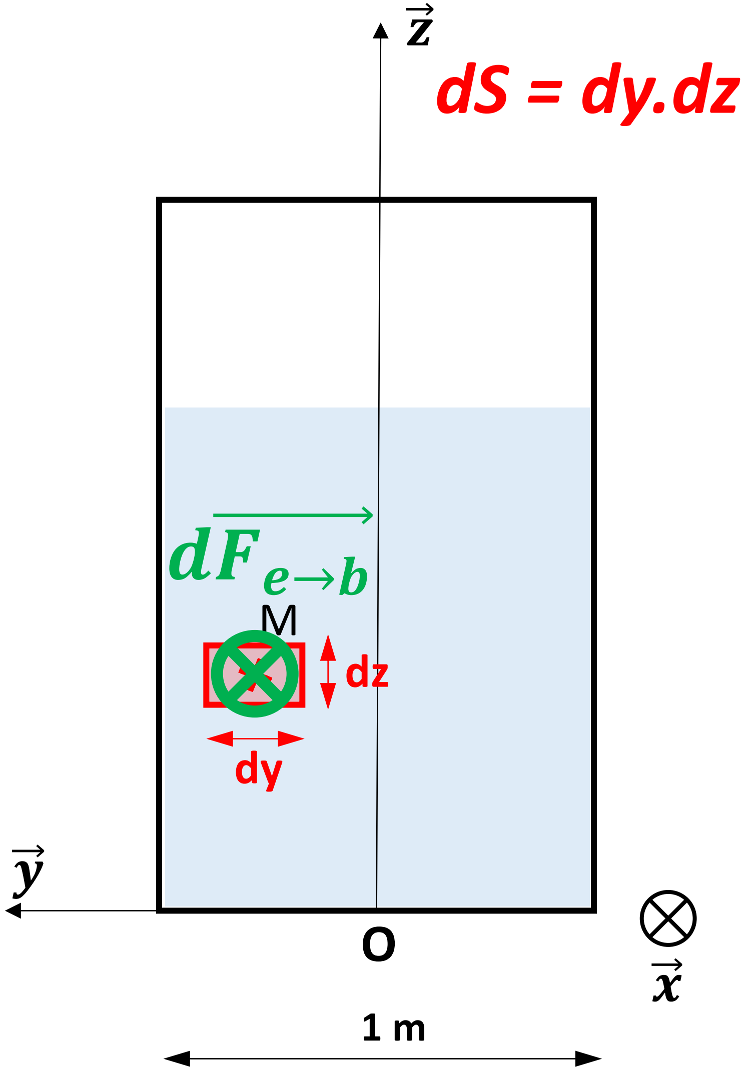
$$dS = dy.dz$$



EXERCICE 2 : Barrage pesant

$$\{dT_{e\to b}\} = \begin{Bmatrix} p(z)dS.\vec{x} \\ \vec{0} \end{Bmatrix}_M$$

$$\{T_{e\to b}\} = \begin{Bmatrix} \vec{R}_{e\to b} = \int_S p(z)dS.\vec{x} \\ \vec{M}_{O,e\to b} = \int_S \overrightarrow{OM} \wedge p(z)dS.\vec{x} \end{Bmatrix}_O$$



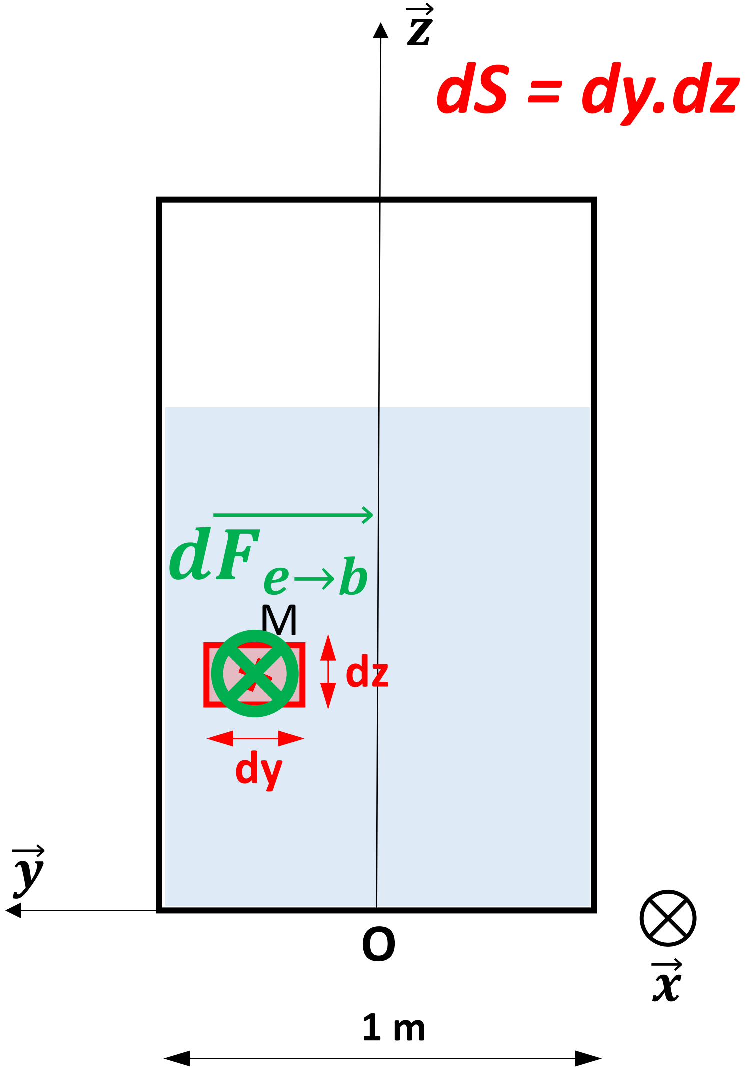
EXERCICE 2 : Barrage pesant

$$\overrightarrow{R}_{e\to b} = \int_S p(z)dS.\vec{x}$$

$$\overrightarrow{R}_{e\to b} = \int_{z=0}^{h}\int_{y=-1/2}^{1/2}\left(p_0 + \rho g(h-z)\right)dydz.\vec{x}$$

$$\overrightarrow{R}_{e\to b} = \int_{z=0}^{h}\left(p_0 + \rho g(h-z)\right)dz.\vec{x}$$

$$\boxed{\overrightarrow{R}_{e\to b} = \left(p_0 h + \frac{\rho g h^2}{2}\right).\vec{x}}$$



EXERCICE 2 : Barrage pesant

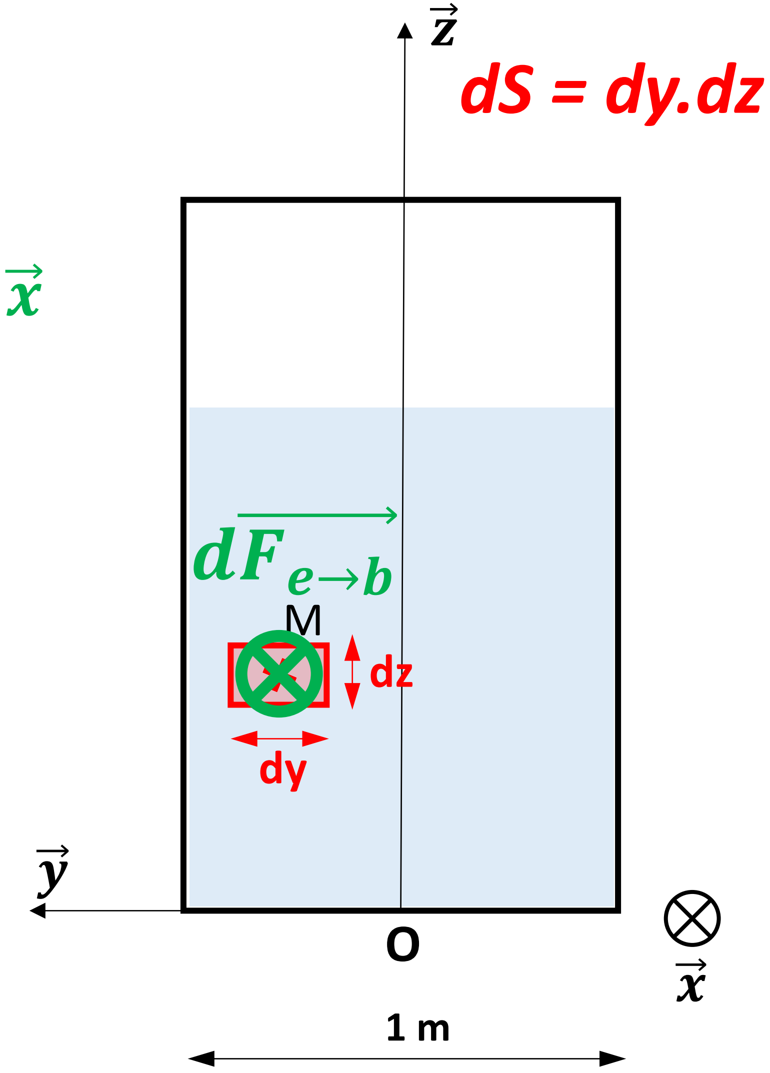
$$\overrightarrow{M}_{O,e\to b} = \int_S \overrightarrow{OM} \wedge p(z)dS.\vec{x}$$

$$\overrightarrow{M}_{O,e\to b} = \int_{z=0}^{h} \int_{y=-1/2}^{1/2} (y.\vec{y} + z.\vec{z}) \wedge \left(p_0 + \rho g(h-z)\right)dydz.\vec{x}$$

$$\overrightarrow{M}_{O,e\to b} = \int_{z=0}^{h} \int_{y=-1/2}^{1/2} (-y.\vec{z} + z.\vec{y})\left(p_0 + \rho g(h-z)\right)dydz$$

$$\overrightarrow{M}_{O,e\to b} = \int_{z=0}^{h} \int_{y=-1/2}^{1/2} z.\vec{y}.\left(p_0 + \rho g(h-z)\right)dydz$$

$$\boxed{\overrightarrow{M}_{O,e\to b} = \left(p_0\frac{h^2}{2} + \frac{\rho g h^3}{6}\right).\vec{y}}$$



EXERCICE 2 : Barrage pesant

Position du centre de poussée P :

P est tel que le torseur des actions mécaniques de l'eau sur le barrage est réductible à un glisseur ⟺ torseur dont le moment est nul

Notons $\overrightarrow{OP} = y_P.\vec{y} + z_P.\vec{z}$

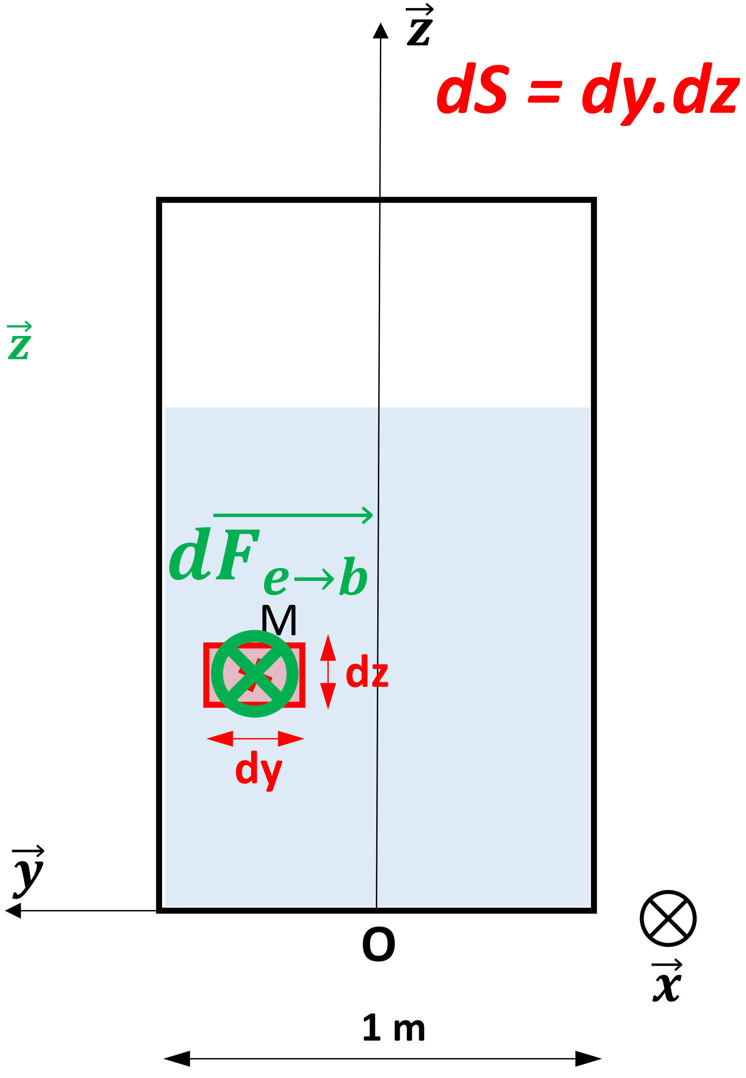
$$\Leftrightarrow \overrightarrow{M_{P,e\to b}} = \vec{0}$$

$$\Leftrightarrow \overrightarrow{M_{O,e\to b}} + \overrightarrow{PO} \wedge \overrightarrow{R_{e\to b}} = \vec{0}$$

$$\Leftrightarrow \left(p_0\frac{h^2}{2} + \frac{\rho g h^3}{6}\right).\vec{y} - (y_P.\vec{y} + z_P.\vec{z}) \wedge \left(p_0 h + \frac{\rho g h^2}{2}\right).\vec{x} = \vec{0}$$

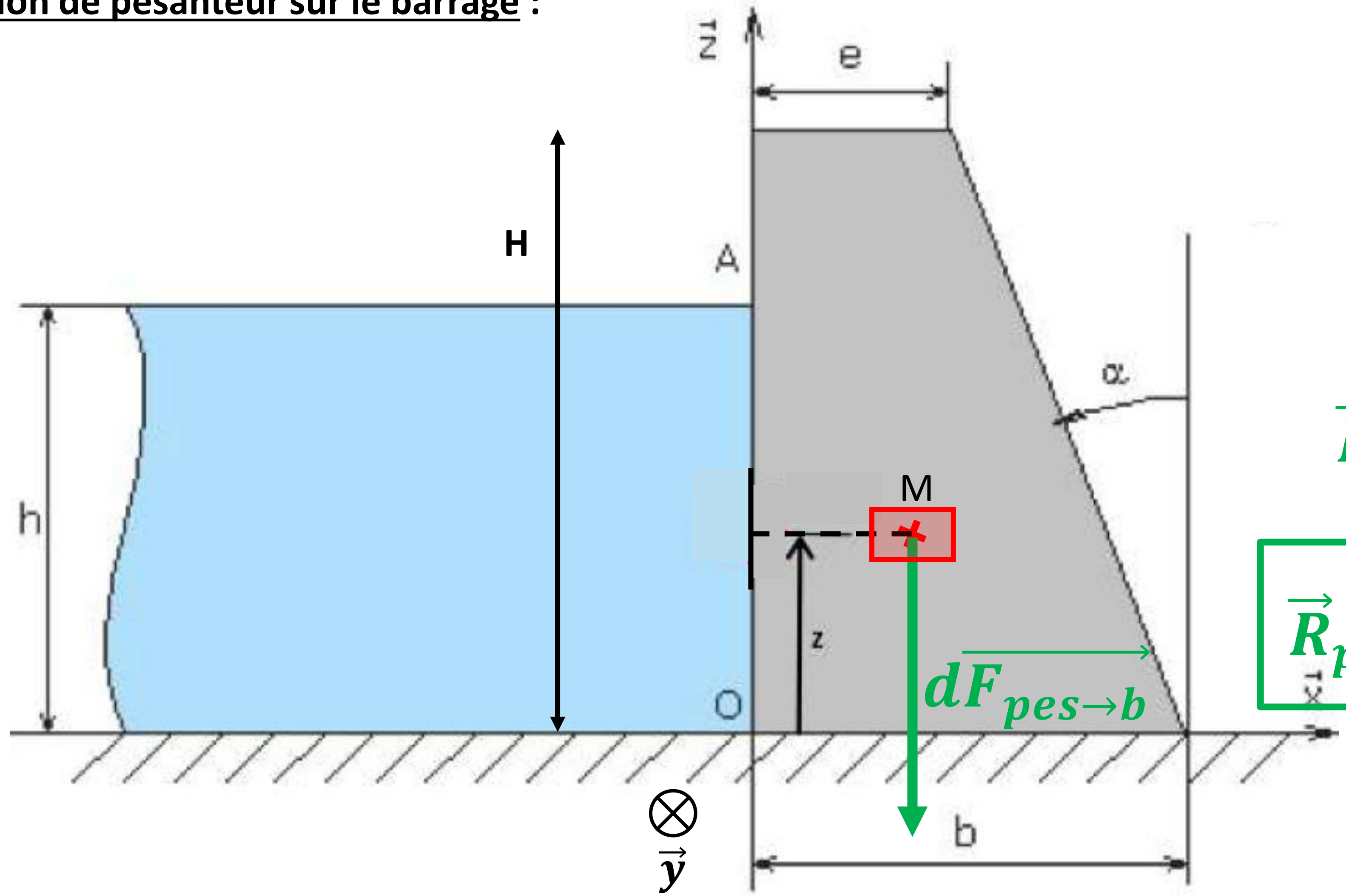
$$\Leftrightarrow \left(p_0\frac{h^2}{2} + \frac{\rho g h^3}{6}\right).\vec{y} - (-y_P.\vec{z} + z_P.\vec{y}).\left(p_0 h + \frac{\rho g h^2}{2}\right) = \vec{0}$$

$$\Leftrightarrow \begin{cases} y_P = 0 \\ z_P = \dfrac{p_0\dfrac{h}{2} + \dfrac{\rho g h^2}{6}}{p_0 + \dfrac{\rho g h}{2}} \end{cases}$$



EXERCICE 2 : Barrage pesant

Action de pesanteur sur le barrage :



$$d\overrightarrow{F_{pes\to S}} = -\mu g\vec{z}dV$$

μ : masse volumique [kg.m^{-3}]

$dV = dx.dy.dz$

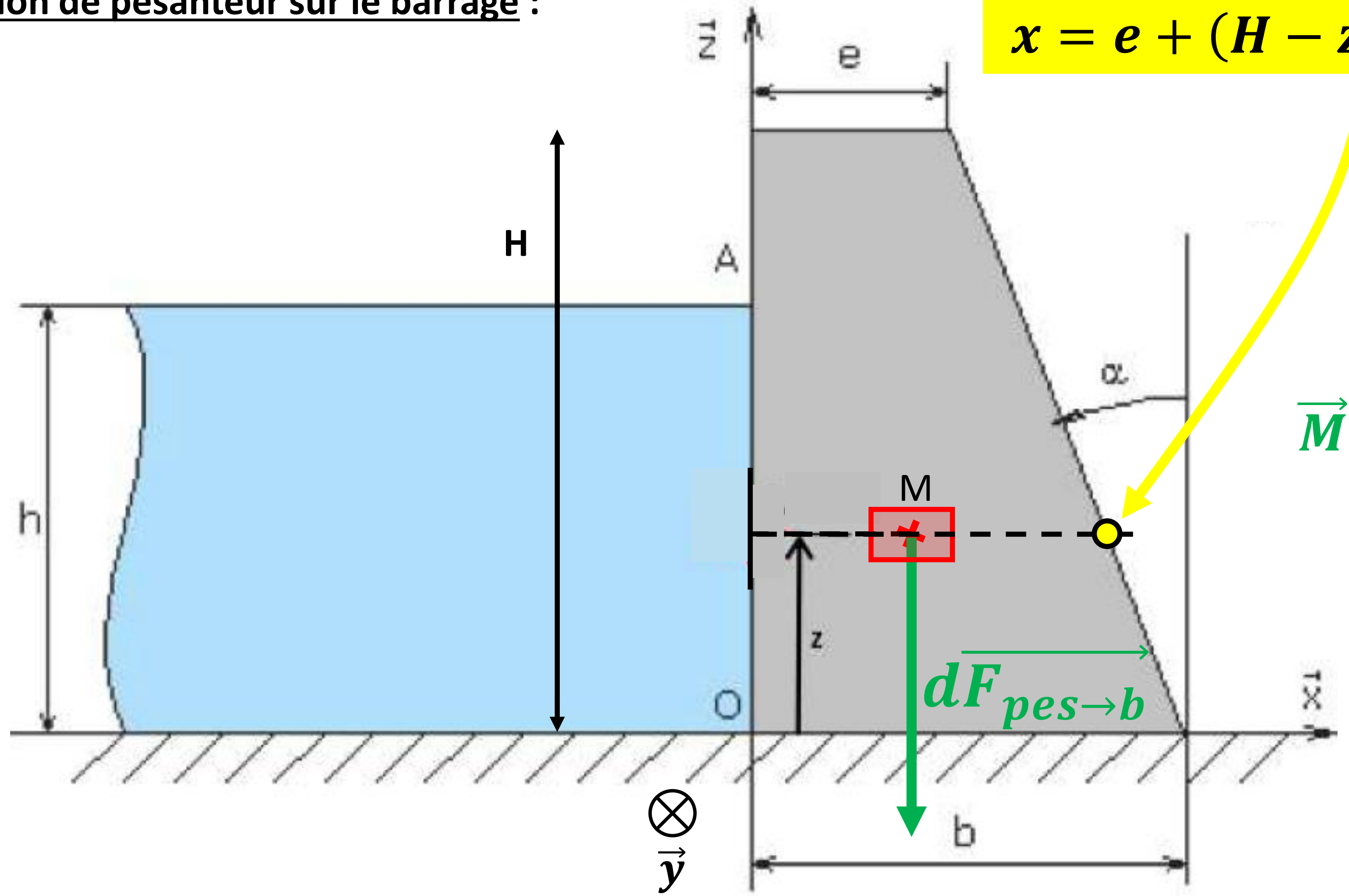
$$\overrightarrow{R_{pes\to S}} = -\mu g.\vec{z}\int_S dV$$

$$\overrightarrow{R_{pes\to S}} = -\mu g\frac{(b+e)H}{2}.\vec{z}$$

EXERCICE 2 : Barrage pesant

Action de pesanteur sur le barrage :

$$x = e + (H - z)\tan\alpha$$



$$\overrightarrow{dF_{pes\to S}} = -\mu g\vec{z}dV$$

μ : masse volumique [kg.m^{-3}]

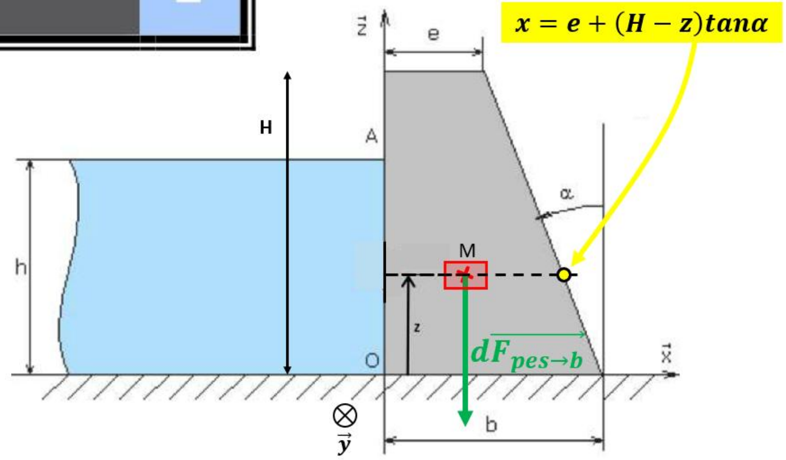
$dV = dx.dy.dz$

$$\overrightarrow{M_{O,pes\to S}} = \int_S \overrightarrow{OM} \wedge \overrightarrow{dF_{pes\to S}}$$

$$= \int_S \overrightarrow{OM} \wedge -\mu g\vec{z}.dV$$

$$\overrightarrow{OM} = x.\vec{x} + y.\vec{y} + z.\vec{z}$$

EXERCICE 2 : Barrage pesant



$$\overrightarrow{M}_{O,pes\to S} = \int_S \overrightarrow{OM} \wedge \overrightarrow{dF_{pes\to S}}$$

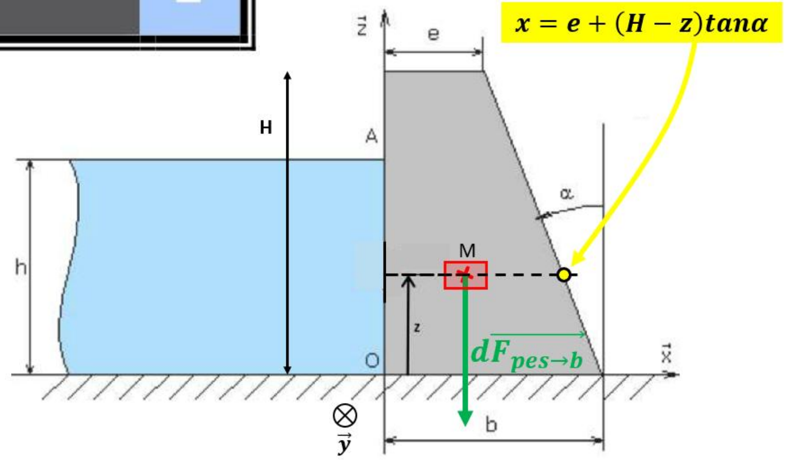
$$\overrightarrow{M}_{O,pes\to S} = -\mu g \int_S (x.\vec{x} + y.\vec{y} + z.\vec{z}) \wedge \vec{z}.dx.dy.dz$$

$$\overrightarrow{M}_{O,pes\to S} = -\mu g \int_S (y.\vec{x} - x.\vec{y})dx.dy.dz$$

$$\overrightarrow{M}_{O,pes\to S} = -\mu g \int_{z=0}^{H} \int_{y=-1/2}^{1/2} \int_{x=0}^{e+(H-z)tan\alpha} (y.\vec{x} - x.\vec{y})dx.dy.dz$$

$$\overrightarrow{M}_{O,pes\to S} = \mu g \int_{z=0}^{H} \int_{x=0}^{e+(H-z)tan\alpha} x.\vec{y}.dx.dz$$

EXERCICE 2 : Barrage pesant



$$\overrightarrow{M_{O,pes \to S}} = \mu g \int_{z=0}^{H} \int_{x=0}^{e+(H-z)tan\alpha} x.\vec{y}.dx.dz$$

$$\overrightarrow{M_{O,pes \to S}} = \frac{1}{2}\mu g \vec{y} \int_{z=0}^{H} [e + (H - z)tan\alpha]^2 dz$$

$$\overrightarrow{M_{O,pes \to S}} = \frac{1}{2}\mu g \vec{y} \left[-\frac{1}{3tan\alpha}(e + (H - z)tan\alpha)^3 \right]_0^H$$

$$\overrightarrow{M_{O,pes \to S}} = \frac{1}{2}\mu g \vec{y} \left[-\frac{1}{3tan\alpha}(e + (H - z)tan\alpha)^3 \right]_0^H$$

$$\overrightarrow{M_{O,pes \to S}} = -\frac{1}{6tan\alpha}\mu g\left(e^3 - (e + Htan\alpha)^3\right)\vec{y}$$

$$\{T_{pes \to S}\} = \left\{ \begin{matrix} -\mu g \dfrac{(b+e)H}{2}.\vec{z} \\ \dfrac{b^3 - e^3}{6tan\alpha}\mu g \vec{y} \end{matrix} \right\}_O$$

$$\overrightarrow{M_{O,pes \to S}} = \frac{b^3 - e^3}{6tan\alpha}\mu g \vec{y}$$